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The eigenvalue problem of a specially updated matrix

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Abstract

We study the eigenvalue problem for a specially structured rank-k updated matrix, based on the Sherman–Morrison–Woodbury formula.

Keywords: Rank-k updated matrix; Sherman–Morrison–Woodbury

1. Introduction

In a recent paper [2], the eigenvalue problem of a special rank-one updated matrix was studied and the result therein provided an alternative proof of the eigenvalue theorem [3–6] for the Google matrix whose eigenvector associated with eigenvalue 1 is the so-called PageRank for the Google web search engine [1,7,8]. The main result of [2] is the following theorem.

Theorem 1.1. *Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ counting algebraic multiplicities, and let u and v be two n-dimensional real column vectors such that v is a left eigenvector of A associated with eigenvalue λ_1. Then, the eigenvalues of the matrix*

$$B = A + uv^{\mathrm{T}}$$

are

$$\{\lambda_1 + u^{\mathrm{T}}v, \lambda_2, \lambda_3, \ldots, \lambda_n\}.$$

In this paper, we generalize Theorem 1.1 by considering the eigenvalue problem of the matrix

$$B = A + u_1v_1^{\mathrm{T}} + u_2v_2^{\mathrm{T}} + \cdots + u_kv_k^{\mathrm{T}}$$

with $2 \leqslant k \leqslant n$, where $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ are real column vectors such that $v_1, \ldots, v_k$ are left eigenvectors of A.

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Instead of the Sherman–Morrison formula used in [2] for the inverse of a rank-one updated matrix, we need the following Sherman–Morrison–Woodbury formula for the inverse of a rank-k updated matrix for our purpose.

Lemma 1.1. *If A is an invertible $n \times n$ real matrix, and U, V are two $n \times k$ real matrices, then the $n \times n$ matrix $A + UV^{\mathrm{T}}$ is invertible if and only if the $k \times k$ matrix $I + V^{\mathrm{T}}A^{-1}U$ is invertible, and then*

$$(A + UV^{\mathrm{T}})^{-1} = A^{-1} - A^{-1}U(I + V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}. \tag{1}$$

Although the proof of our main result Theorem 3.1 works for any $k \leqslant n$, in the next section we prove the special case $k = 2$ first to illustrate the basic idea of our approach. Then we give the general result in Section 3.

2. Eigenvalues of rank-2 updated matrices

We first consider the special case $k = 2$ in this section. Let A be an $n \times n$ real matrix and let u_1, u_2, v_1, v_2 be n-dimensional real column vectors. We consider the eigenvalue problem for the matrix

$$B = A + u_1v_1^{\mathrm{T}} + u_2v_2^{\mathrm{T}}.$$

In the proof of our theorems below, we use the standard notation in matrix theory. For example, $N(A)$ denotes the null space of A and $M^{\perp}$ is the orthogonal complement of a subset M in R^n.

Theorem 2.1. *Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ counting algebraic multiplicities, and let u_1, u_2, v_1, v_2 be real column vectors such that v_1 and v_2 are linearly independent left eigenvectors of A corresponding to eigenvalues λ_1 and λ_2, respectively. Then the eigenvalues of the matrix $B = A + u_1v_1^{\mathrm{T}} + u_2v_2^{\mathrm{T}}$ are*

$$\{\mu, v, \lambda_3, \ldots, \lambda_n\},$$

where μ and v are the eigenvalues of the 2×2 matrix

$$W = \begin{bmatrix} \lambda_1 + u_1^{\mathrm{T}}v_1 & u_1^{\mathrm{T}}v_2 \\ u_2^{\mathrm{T}}v_1 & \lambda_2 + u_2^{\mathrm{T}}v_2 \end{bmatrix} = \mathrm{diag}(\lambda_1, \lambda_2) + U^{\mathrm{T}}V \tag{2}$$

with U and V denoting the $n \times 2$ matrices

$$U = [u_1, u_2], \quad V = [v_1, v_2].$$

Proof. For any complex number λ,

$$B - \lambda I = A - \lambda I + UV^{\mathrm{T}}. \tag{3}$$

We first show that μ and v are eigenvalues of B. Since $v_1^{\mathrm{T}}A = \lambda_1v_1^{\mathrm{T}}$ and $v_2^{\mathrm{T}}A = \lambda_2v_2^{\mathrm{T}}$, we have

$$v_1^{\mathrm{T}}B = v_1^{\mathrm{T}}A + v_1^{\mathrm{T}}u_1v_1^{\mathrm{T}} + v_1^{\mathrm{T}}u_2v_2^{\mathrm{T}} = (\lambda_1 + u_1^{\mathrm{T}}v_1)v_1^{\mathrm{T}} + (u_2^{\mathrm{T}}v_1)v_2^{\mathrm{T}}$$

and

$$v_2^{\mathrm{T}}B = v_2^{\mathrm{T}}A + v_2^{\mathrm{T}}u_1v_1^{\mathrm{T}} + v_2^{\mathrm{T}}u_2v_2^{\mathrm{T}} = (u_1^{\mathrm{T}}v_2)v_1^{\mathrm{T}} + (\lambda_2 + u_2^{\mathrm{T}}v_2)v_2^{\mathrm{T}}.$$

That is,

$$V^{\mathrm{T}}B = W^{\mathrm{T}}V^{\mathrm{T}}, \tag{4}$$

where W is the 2×2 matrix as defined by (2). Suppose λ is an eigenvalue of W. Then there is a nonzero 2-dimensional column vector η such that $\eta^{\mathrm{T}}(W^{\mathrm{T}} - \lambda I) = 0$, which and (4) imply that

$$\eta^{\mathrm{T}}V^{\mathrm{T}}(B - \lambda I) = \eta^{\mathrm{T}}(W^{\mathrm{T}} - \lambda I)V^{\mathrm{T}} = 0.$$

Since v_1, v_2 are linearly independent, $V\eta$ is a left eigenvector of B associated with eigenvalue λ. Therefore, both μ and v are eigenvalues of B.

Next we show that a complex number λ is an eigenvalue of B if and only if $\lambda \in \{\mu, v, \lambda_3, \ldots, \lambda_n\}$. We consider four cases separately.

(i) Assume $\lambda \neq \lambda_i$ for all $i = 1, \ldots, n$. Then the inverse matrix $(A - \lambda I)^{-1}$ exists. If λ is an eigenvalue of B, then

$$\det[I + V^{\mathrm{T}}(A - \lambda I)^{-1}U] = 0 \tag{5}$$

by the Sherman–Morrison–Woodbury formula applied to (3). Since $V^{\mathrm{T}}A = \mathrm{diag}(\lambda_1, \lambda_2)V^{\mathrm{T}}$,

$$V^{\mathrm{T}}(A - \lambda I) = \mathrm{diag}(\lambda_1 - \lambda, \lambda_2 - \lambda)V^{\mathrm{T}},$$

which implies that

$$V^{\mathrm{T}}(A - \lambda I)^{-1} = \mathrm{diag}\left(\frac{1}{\lambda_1 - \lambda}, \frac{1}{\lambda_2 - \lambda}\right)V^{\mathrm{T}}.$$

It follows that

$$\begin{aligned}\det[I + V^{\mathrm{T}}(A - \lambda I)^{-1}U] &= \det\left[I + \mathrm{diag}\left(\frac{1}{\lambda_1 - \lambda}, \frac{1}{\lambda_2 - \lambda}\right)V^{\mathrm{T}}U\right] \\ &= \det\left[\mathrm{diag}\left(\frac{1}{\lambda_1 - \lambda}, \frac{1}{\lambda_2 - \lambda}\right)\right] \cdot \det\left[\mathrm{diag}(\lambda_1, \lambda_2) + U^{\mathrm{T}}V - \lambda I\right] \\ &= \frac{1}{(\lambda_1 - \lambda)(\lambda_2 - \lambda)} \cdot \det(W - \lambda I).\end{aligned}$$

So, equality (5) implies that $\det(W - \lambda I) = 0$. That is, λ is an eigenvalue of the 2×2 matrix W. Therefore $\lambda = \mu$ or v. This proves that all the other eigenvalues of B are inside the eigenvalue set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of A.

(ii) Suppose $\lambda = \lambda_i$ for some $i = 3, \ldots, n$. Then the matrix $A - \lambda I$ is singular. Since

$$A - \lambda I = B - \lambda I - UV^{\mathrm{T}}, \tag{6}$$

if λ_i is not an eigenvalue of B, then the Sherman–Morrison–Woodbury formula applied to (6) implies that

$$\det\left[I - V^{\mathrm{T}}(B - \lambda I)^{-1}U\right] = 0. \tag{7}$$

Since $V^{\mathrm{T}}(B - \lambda I) = (W^{\mathrm{T}} - \lambda I)V^{\mathrm{T}}$ from (4),

$$V^{\mathrm{T}}(B - \lambda I)^{-1} = (W^{\mathrm{T}} - \lambda I)^{-1}V^{\mathrm{T}}.$$

Thus,

$$\begin{aligned}\det[I - V^{\mathrm{T}}(B - \lambda I)^{-1}U] &= \det[I - (W^{\mathrm{T}} - \lambda I)^{-1}V^{\mathrm{T}}U] = \det[(W^{\mathrm{T}} - \lambda I)^{-1}] \cdot \det[\mathrm{diag}(\lambda_1 - \lambda, \lambda_2 - \lambda)] \\ &= \frac{(\lambda_1 - \lambda)(\lambda_2 - \lambda)}{\det(W - \lambda I)}.\end{aligned}$$

It follows from (7) that $\lambda = \lambda_1$ or λ_2, which is a contradiction if $\lambda_i \neq \lambda_1, \lambda_2$. This says that λ_i is an eigenvalue of B under the additional assumption that $\lambda_i \neq \lambda_1, \lambda_2$.

(iii) Now suppose $\lambda = \lambda_i = \lambda_1$ for some $i = 3, \ldots, n$. Then the algebraic multiplicity of the eigenvalue λ for A is at least two. First assume that $\dim N(A - \lambda I) < \dim N[(A - \lambda I)^2]$. Then there is a nonzero vector $u \in N[(A - \lambda I)^2]$ such that $v \equiv (A - \lambda I)u \neq 0$. Since $(\lambda_2 - \lambda_1)^2 v_2^{\mathrm{T}}u = v_2^{\mathrm{T}}(A - \lambda I)^2 u = 0$, there holds $(\lambda_2 - \lambda_1)v_2^{\mathrm{T}}u = 0$. Therefore, by (3),

$$\begin{aligned}(B - \lambda I)v &= (B - \lambda I)(A - \lambda I)u = [(A - \lambda I)^2 + UV^{\mathrm{T}}(A - \lambda I)]u = (A - \lambda I)^2 u + (\lambda_2 - \lambda_1)u_2 v_2^{\mathrm{T}}u \\ &= (\lambda_2 - \lambda_1)v_2^{\mathrm{T}}u \cdot u_2 = 0.\end{aligned}$$

That is, λ is an eigenvalue of B with eigenvector v. Next assume that $\dim N(A-\lambda I)=\dim N[(A-\lambda I)^2]\geqslant 2$. If $\lambda_1=\lambda_2$, then $\dim N(A-\lambda I)\geqslant 3$. Since $\dim\{v_1,v_2\}^\perp=n-2$, $\dim N(B-\lambda I)\geqslant 1$ by (3). In other words, λ is an eigenvalue of B. If $\lambda_1\neq\lambda_2$, then there is $u\neq 0$ such that $(A-\lambda I)u=0$ and $v_1^{\mathrm{T}}u=0$ since $\dim\{v_1\}^\perp=n-1$. Since $(\lambda_2-\lambda_1)v_2^{\mathrm{T}}u=v_2^{\mathrm{T}}(A-\lambda I)u=0$, we also have $v_2^{\mathrm{T}}u=0$. Hence,

$$(B-\lambda I)u=(A-\lambda I)u+UV^{\mathrm{T}}u=0.$$

That is, λ is an eigenvalue of B with eigenvector u. By the same token, if $\lambda_i=\lambda_2$ for some $i=3,\ldots,n$, then λ_i is an eigenvalue of B.

(iv) Finally, we show that for $\lambda=\lambda_1$ or λ_2, if $\lambda\neq\lambda_i$ for all $i=3,\ldots,n$, and if λ is not an eigenvalue of W, then λ is not an eigenvalue of B. We prove the claim for $\lambda=\lambda_1$ only since the proof for the other case is exactly the same. What we need to show is that the matrix $B-\lambda I$ is nonsingular. Let $w\in R^n$ be a nonzero vector. First assume that $w=V\eta$ for some nonzero vector $\eta\in R^2$. Since $\lambda\neq\mu$, v, the 2×2 matrix $W-\lambda I$ is nonsingular, so $\eta^{\mathrm{T}}(W^{\mathrm{T}}-\lambda I)\neq 0$. Since the rank of V is 2, equality (4) gives

$$w^{\mathrm{T}}(B-\lambda I)=\eta^{\mathrm{T}}V^{\mathrm{T}}(B-\lambda I)=\eta^{\mathrm{T}}(W^{\mathrm{T}}-\lambda I)V^{\mathrm{T}}\neq 0.$$

Now assume that $w\notin\operatorname{span}\{v_1,v_2\}$. Suppose $w^{\mathrm{T}}(B-\lambda I)=w^{\mathrm{T}}(A-\lambda I)+w^{\mathrm{T}}UV^{\mathrm{T}}=0$. Then $w^{\mathrm{T}}(A-\lambda_1 I)=-w^{\mathrm{T}}UV^{\mathrm{T}}$, so

$$w^{\mathrm{T}}(A-\lambda_2 I)(A-\lambda_1 I)^2=-w^{\mathrm{T}}UV^{\mathrm{T}}(A-\lambda_1 I)(A-\lambda_2 I)=0 \tag{8}$$

since $V^{\mathrm{T}}(A-\lambda_1 I)(A-\lambda_2 I)=0$. Eq. (8) and the fact that the algebraic multiplicity of the eigenvalue λ_1 of A is at most 2 imply that $w^{\mathrm{T}}(A-\lambda_2 I)=\eta^{\mathrm{T}}V^{\mathrm{T}}$ for some $\eta\in R^2$. Therefore,

$$\eta^{\mathrm{T}}V^{\mathrm{T}}=w^{\mathrm{T}}(A-\lambda_2 I)=w^{\mathrm{T}}(A-\lambda_1 I)+(\lambda_1-\lambda_2)w^{\mathrm{T}}=-w^{\mathrm{T}}UV^{\mathrm{T}}+(\lambda_1-\lambda_2)w^{\mathrm{T}}. \tag{9}$$

If $\lambda_1=\lambda_2$, then $w^{\mathrm{T}}(A-\lambda_1 I)^2=0$ by (9), which contradicts the fact that $\dim N[(A-\lambda_1 I)^2]=2$. If $\lambda_1\neq\lambda_2$, then (9) gives

$$w=\frac{V(\eta+U^{\mathrm{T}}w)}{\lambda_1-\lambda_2},$$

which contradicts the assumption that $w\notin\operatorname{span}\{v_1,v_2\}$. This concludes the proof of the theorem. □

Remark 2.1. Actually, in case (iv) of the above proof, since the algebraic multiplicity of λ_1 is at most 2, by the theory of Jordan forms for matrices, if $w\notin\operatorname{span}\{v_1,v_2\}$, then (8) implies that $w^{\mathrm{T}}(A-\lambda_2 I)(A-\lambda_1 I)^2\neq 0$. This observation will be used later to shorten the proof of Theorem 3.1.

An immediate consequence of Theorem 2.1 is

Corollary 2.1. *Suppose in addition that $u_1^{\mathrm{T}}v_2=0$ and $u_2^{\mathrm{T}}v_1=0$. Then the eigenvalues of B are*

$$\{\lambda_1+u_1^{\mathrm{T}}v_1,\lambda_2+u_2^{\mathrm{T}}v_2,\lambda_3,\ldots,\lambda_n\}.$$

3. Eigenvalues of rank-k updated matrices

The same idea as used in the proof of Theorem 2.1 can be applied to establishing the following general result.

Theorem 3.1. *Let A be an $n\times n$ real matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ counting algebraic multiplicities, and for $1\leqslant k\leqslant n$ let $u_1,\ldots,u_k$ and $v_1,\ldots,v_k$ be real column vectors such that $v_1,\ldots,v_k$ are linearly independent left eigenvectors of A corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_k$, respectively. Then the eigenvalues of the matrix $B=A+\sum_{i=1}^k u_iv_i^{\mathrm{T}}$ are*

$$\{\mu_1,\mu_2,\ldots,\mu_k,\lambda_{k+1},\ldots,\lambda_n\},$$

where $\mu_1,\ldots,\mu_k$ are the eigenvalues of the $k\times k$ matrix

$$W = \operatorname{diag}(\lambda_1, \ldots, \lambda_k) + U^{\mathrm{T}} V \tag{10}$$

and

$$U = [u_1, \ldots, u_k], \quad V = [v_1, \ldots, v_k].$$

Proof. The special cases $k = 1$ and $k = 2$ have been covered by Theorems 1.1 and 2.1, respectively, so we assume $k \geqslant 2$. Exactly the same process in the proof of Theorem 2.1 can be used again to show that

(i) $\mu_1, \ldots, \mu_k$ are eigenvalues of B,
(ii) if $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then λ is not an eigenvalue of B except for $\lambda = \mu_j$ for some $j = 1, \ldots, k$, and
(iii) λ_i is an eigenvalue of B for $i = k+1, \ldots, n$ if $\lambda_i \neq \lambda_j$ for all $j = 1, \ldots, k$.

Now suppose $\lambda_i = \lambda_j$ for some $k+1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$. We show that λ_j is an eigenvalue of B. We just prove the case $j = 1$ since the proof for $j = 2, \ldots, k$ is exactly the same. Without loss of generality, we may assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_l$ and $\lambda_1 \neq \lambda_j$ for all $j = l+1, \ldots, k$. Then, since $\lambda_1 = \lambda_i$, the algebraic multiplicity of the eigenvalue λ_1 for A is at least $l+1$. Let $u \in N[(A - \lambda_1 I)^2]$. Since $v_j^{\mathrm{T}}(A - \lambda_1 I)^2 = (\lambda_j - \lambda_1)^2 v_j^{\mathrm{T}}$,

$$v_j^{\mathrm{T}} u = \frac{1}{(\lambda_j - \lambda_1)^2} v_j^{\mathrm{T}} (A - \lambda_1 I)^2 u = 0, \quad j = l+1, \ldots, k. \tag{11}$$

Suppose first that $\dim N(A - \lambda_1 I) < \dim N[(A - \lambda_1 I)^2]$. Then there is $u \in N[(A - \lambda_1 I)^2]$ such that $v \equiv (A - \lambda_1 I)u \neq 0$. Since $v_j^{\mathrm{T}}(A - \lambda_1 I) = 0$ for $j = 1, \ldots, l$ and $v_j^{\mathrm{T}}(A - \lambda_1 I) = (\lambda_j - \lambda_1) v_j^{\mathrm{T}}$ for $j = l+1, \ldots, k$, we have

$$(B - \lambda_1 I)(A - \lambda_1 I) = (A - \lambda_1 I)^2 + UV^{\mathrm{T}}(A - \lambda_1 I) = (A - \lambda_1 I)^2 + \sum_{j=l+1}^{k} (\lambda_j - \lambda_1) u_j v_j^{\mathrm{T}}.$$

The above equality and (11) imply that $(B - \lambda_1 I)v = 0$. That is, λ_1 is an eigenvalue of B with eigenvector v.

Now suppose $\dim N(A - \lambda_1 I) = \dim N[(A - \lambda_1 I)^2]$. Then $N(A - \lambda_1 I) = N[(A - \lambda_1 I)^2]$ with dimension at least $l+1$. Since $\dim\{v_1, \ldots, v_l\}^{\perp} = n - l$, there is a nonzero vector $u \in N(A - \lambda_1 I)$ such that $u \in \{v_1, \ldots, v_l\}^{\perp}$. Moreover, by (11), $u \in \{v_{l+1}, \ldots, v_k\}^{\perp}$. Therefore,

$$(B - \lambda_1 I)u = (A - \lambda_1 I)u + UV^{\mathrm{T}} u = 0.$$

So λ_1 is an eigenvalue of B with eigenvector u.

Finally we show that if for some $j = 1, \ldots, k$, λ_j is not an eigenvalue of W defined by (10) and $\lambda_j \neq \lambda_i$ for all $i = k+1, \ldots, n$, then λ_j is not an eigenvalue of B. We assume $j = 1$ for the sake of simplicity of notation.

First let $w = V\eta$ for some nonzero vector $\eta \in R^k$. Since $W - \lambda_1 I$ is nonsingular, $\eta^{\mathrm{T}}(W^{\mathrm{T}} - \lambda_1 I) \neq 0$. The assumption that the rank of V is k and equality (10) give

$$w^{\mathrm{T}}(B - \lambda_1 I) = \eta^{\mathrm{T}} V^{\mathrm{T}} (B - \lambda_1 I) = \eta^{\mathrm{T}} (W^{\mathrm{T}} - \lambda_1 I) V^{\mathrm{T}} \neq 0.$$

Next assume $w \notin \operatorname{span}\{v_1, \ldots, v_k\}$. Suppose $w^{\mathrm{T}}(B - \lambda_1 I) = w^{\mathrm{T}}(A - \lambda_1 I) + w^{\mathrm{T}} UV^{\mathrm{T}} = 0$. Then $w^{\mathrm{T}}(A - \lambda_1 I) = -w^{\mathrm{T}} UV^{\mathrm{T}}$, so

$$w^{\mathrm{T}} \prod_{j=2}^{k} (A - \lambda_j I) \cdot (A - \lambda_1 I)^2 = -w^{\mathrm{T}} UV^{\mathrm{T}} \prod_{j=1}^{k} (A - \lambda_j I) = 0 \tag{12}$$

since $V^{\mathrm{T}} \prod_{j=1}^{k} (A - \lambda_j I) = 0$. On the other hand, since $\lambda_1 \neq \lambda_i$ for all $i = k+1, \ldots, n$, the condition $w \notin \operatorname{span}\{v_1, \ldots, v_k\}$ implies that $w^{\mathrm{T}} \prod_{j=2}^{k} (A - \lambda_j I) \cdot (A - \lambda_1 I)^2 \neq 0$. This gives a contradiction to (12). □

In particular, we have

Corollary 3.1. *Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ counting algebraic multiplicities. For $i = 1, 2, \ldots, k$ let u_i and v_i be n-dimensional real column vectors such that v_i's are linearly independent left eigenvectors of A associated with eigenvalues λ_i respectively. If $u_i^{\mathrm{T}} v_j = 0$ for all $i \neq j$, then the eigenvalues of the matrix*

$$A + \sum_{i=1}^{k} u_i v_i^{\mathrm{T}}$$

are

$$\{\lambda_1 + u_1^{\mathrm{T}} v_1, \ldots, \lambda_k + u_k^{\mathrm{T}} v_k, \lambda_{k+1}, \ldots, \lambda_n\}.$$

Remark 3.1. We point out that the assumption that $v_1, \ldots, v_k$ are linearly independent is not necessary and can be removed from the fact that eigenvalues of a matrix are continuous functions of its entries and any square matrix is a limit of a sequence of matrices with all distinct eigenvalues so that their corresponding eigenvectors are linearly independent.

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